Asymptotic Solution for Pressurized Noncircular Cylinders with Nonuniform Rings

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An asymptotic expansion procedure is used to analyze hydrostatically loaded noncircular cylindrical shells. The asymptotic solution is combined with a solution for singly symmetric oval rings with variable cross-sectional properties to obtain the solution for pressurized noncircular cylinders with nonuniform rings. Numerical results indicate that the maximum stress in the ring-shell structure can be significantly reduced by appropriately varying the ring depth while keeping the total volume constant.

Nomenclature

A^*	= area enclosed by oval
A, A_j, B_j, C_j, b_0	= complex functions of s, see Eqs. (44, 28a–c, and 38), respectively
C	= $[3(1-\nu^2)]^{-1/2}$
c_1, c_2, c_3	= constants
d, f	= depth and flange width of tee-section ring, respectively
E^*	= Young's modulus of shell
e_x, e_s, e_{sx}	= axial, circumferential, and shear strains of the median surface of the shell
$\tilde{e}^P$	= function defined by Eq. (33c)
$F_0, \ldots F_4$	= functions of s defined by Eqs. (64)
$f_0, \ldots f_6$	= functions of s, see Eqs. (61)
h	= shell thickness
i	= $(-1)^{1/2}$
J	= integral operator defined by Eq. (65)
j, k	= integers
K	= parameter defined by Eq. (4b)
L_0, L	= circumferential and axial lengths of the shell, respectively
M_x, M_s, M_{sx}	= axial, circumferential, and twisting moment resultants, respectively, in the shell
N_x, N_s, N_{sx}	= axial, circumferential, and in-plane shear stress resultants, respectively, in the shell
$\tilde{N}, \tilde{Q}$	= functions defined by Eqs. (33a,b), respectively
P	= function defined by Eq. (59)
$P_x, P_s, P_{sx}, R_x, R_s, R_{sx}$	= complex functions defined by Eqs. (11)
$(Q_x)_{eff}, (N_{sx})_{eff}$	= effective transverse shear and in-plane shear in shell
Q_R, T_R	= values of $(Q_x)_{eff}$ and $(N_{sx})_{eff}$ at $x = x_R$, respectively
q_0	= uniform hydrostatic pressure
r	= local radius of curvature of reference line
r_0	= $L_0/2\pi$
S, Z	= shear and radial interaction forces acting upon reference line of ring, respectively
S_T, Z_T	= total shear and radial forces acting on reference line of the ring, respectively
t, t_f	= thickness of web and flange of ring, respectively
u, v, w	= axial, circumferential, and radially inward displacement components of any point in the shell
V, W	= circumferential and radially inward displacement components of a point on the reference line of the ring, respectively
$\tilde{v}^P$	= function of s, see Eqs. (32)
x, s, z	= axial, circumferential, and radial coordinates of a point in the shell
x_R	= $L/2r_0$
$\bar{z}_c$	= location of centroid of ring
$\bar{z}_0$	= value of $\bar{z}_c$ for tee-section of uniform depth
α, β	= axial coordinates equal to Kx and x/K, respectively
δ	= tracer constant
ϵ	= strain of a fiber along the reference line of the ring
ζ_j	= parameter used to specify the variation of $\bar{z}_c$, see Eq. (69)
θ	= angle between axis of symmetry and outer normal to reference line
$\kappa_x, \kappa_s, \kappa_{sx}$	= curvature parameters of shell
λ	= complex function of s defined by Eq. (45)
ν	= Poisson's ratio
ξ	= parameter that fixes the eccentricity of oval
ρ	= r/r_0
$\sigma_x, \sigma_s, \sigma_{sx}$	= axial, circumferential, and in-plane shear stresses, respectively, in the shell
Φ, Ψ	= complex functions defined by Eqs. (14)
φ, ψ	= imaginary parts of Φ and Ψ, respectively
$(\ )^*$	= dimensional counterpart of ( )
$(\ )^P, (\ )^i, (\ )^e$	= particular, interior, and edge part of ( )
$\overline{(\ )}$	= complex conjugate of ( )
$(\ )'$	= $d(\ )/ds$
$(\ )_{,\alpha}$	= $\partial(\ )/\partial\alpha$
$Re(\ ), Im(\ )$	= real and imaginary parts of ( ), respectively

Introduction

INVESTIGATIONS of the design and fabrication of submergence vehicles has led to a study of noncircular ring-reinforced cylindrical shells. In this connection, the case of a noncircular cylinder with uniform reinforcing rings was analyzed by both an energy approach,[1] in which energy solutions for an oval cylinder[2] and an oval ring[3] were combined, and by an asymptotic series approach.[4] These two approaches were in good agreement with available experimental results.[5,6]

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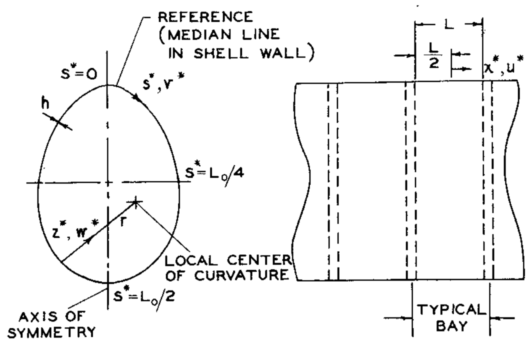

Fig. 1 Ring-reinforced oval cylinder.

In addition to assuming that the cross-sectional properties of both ring and shell are uniform, the theoretical work of Refs. 1–5 assumes that the oval is doubly symmetric and has the special mathematical form discussed in Ref. 7. In the present work an asymptotic expansion procedure is applied to a hydrostatically loaded oval cylindrical shell with nonuniform reinforcing rings. The analysis considers arbitrary singly or doubly symmetric ovals of slow circumferential variation. The shell solution is combined with an appropriate ring solution[8] to obtain the complete solution for a typical bay of a long noncircular cylinder reinforced by many closely spaced nonuniform rings (see Fig. 1). Numerical results indicate that the maximum stress in the ring-shell structure can be significantly reduced by appropriately varying the ring depth while keeping the total volume constant.

The asymptotic expansion procedure, introduced[9] and later extended[10] by E. Reissner, is applicable to thin shells subjected to loads having a slow circumferential variation. More recently, Reissner and Simmonds[11] have removed the restriction of slow circumferential variation. The general procedure[9–11] for obtaining solutions to the governing partial differential equations of thin shell theory is similar in many respects to that proposed by Gol'denveizer.[12]

Governing Equations and Boundary Conditions

In the present analysis, it is assumed that the behavior of the cylindrical shell shown in Fig. 1 can be described by differential equations of the Love type. A tracer is used to keep account of the terms usually omitted in the more simple equations of the Donnell type. These equations can be cast into a complex form which is concise and has the advantage of halving the order of the governing partial differential equations.

In accordance with Love's approximations, the stresses are assumed to vary linearly through the shell wall of uniform thickness h, hence

$$\{\sigma_x,\sigma_s,\sigma_{sx}\} = \{N_x,N_s,N_{sx}\} + (2z^*/h)\{M_x,M_s,M_{sx}\} \quad (1)$$

where z^* is a radially inward coordinate measured from the median surface of the shell. The quantities σ_x, N_x, and M_x represent the total nondimensional axial stress, axial membrane stress, and axial bending stress, respectively. They are related to their corresponding dimensional quantities $\sigma_x{}^*$, $N_x{}^*$, and $M_x{}^*$ by

$$\sigma_x = \sigma_x{}^*/q_0,\ N_x = N_x{}^*/q_0h,\ M_x = M_x{}^*/q_0h^2 \quad (2\text{a,b,c})$$

where q_0 is the applied uniform external pressure. Similar expressions with subscripts s and sx represent circumferential and in-plane shear quantities, respectively (see Fig. 2). Here and in what follows, an asterisk is used to denote a physical rather than a nondimensional quantity.

The nondimensional membrane and bending stresses are related to the median surface strain parameters e_x, e_s, e_{sx}, and curvature parameters $\varkappa_x$, $\varkappa_s$, $\varkappa_{sx}$ by the equations

$$N_x = 3C^2(e_x + \nu e_s),\ N_s = 3C^2(e_s + \nu e_x) \quad (3\text{a,b})$$

$$N_{sx} = N_{xs} = 3C^2(1 - \nu)e_{sx}/2 \quad (3\text{c})$$

$$M_x = -3C(\varkappa_x + \nu\varkappa_s)/2K^2,\ M_s = -3C(\varkappa_s + \nu\varkappa_x)/2K^2 \quad (3\text{d,e})$$

$$M_{sx} = -M_{xs} = -3C(1 - \nu)\varkappa_{sx}/2K^2 \quad (3\text{f})$$

where ν is Poissons' ratio and K and C are defined as

$$C = [3(1 - \nu^2)]^{-1/2},\ K^2C = r_0/h \quad (4\text{a,b})$$

Here, r_0 is the radius of a reference circle whose circumference L_0 is equal to the length of the median line of the shell. The strain and curvature parameters are related to the cooresponding physical quantities as follows:

$$\{e_x,e_s,e_{sx},\varkappa_x,\varkappa_s,\varkappa_{sx}\} = E^*\{e_x{}^*,e_s{}^*,e_{sx}{}^*,r_0\varkappa_x{}^*,r_0\varkappa_s{}^*,r_0\varkappa_{sx}{}^*\}/q_0 \quad (5)$$

where E^* is Young's modulus of the shell.

The median surface strains and curvatures are related to the nondimensional axial, circumferential, and radially inward median surface displacement components u, v, w, respectively, by the expressions

$$e_x = u_{,x},\ e_s = v_{,s} - w/\rho,\ e_{sx} = u_{,s} + v_{,x} \quad (6\text{a,b,c})$$

$$\varkappa_x = w_{,xx},\ \varkappa_s = (w_{,s} + \delta v/\rho)_{,s},\ \varkappa_{sx} = (w_{,s} + \delta v/\rho)_{,x} \quad (6\text{d,e,f})$$

where x and s are nondimensional axial and circumferential median surface coordinates (see Fig. 1), ρ is the nondimensional circumferential local radius of curvature of the shell median surface, δ is a tracer constant which is unity for Love type equations and zero for Donnell type equations, and a comma indicates differentiation with respect to the variable which it precedes. The nondimensional quantities are related to their dimensional counterparts by

$$\{x,s,\rho\} = \{x^*,s^*,r\}/r_0,\ \{u,v,w\} = E^*\{u^*,v^*,w^*\}/q_0r_0 \quad (7\text{a,b})$$

The use of the theorem of the minimum of the total potential together with Eqs. (3) and (6) yields the following equilibrium equations for the shell:

$$N_{x,x} + N_{sx,s} = 0 \quad (8\text{a})$$

$$N_{s,s} + N_{sx,x} - \delta(2M_{sx,x} + M_{s,s})/6\rho K^2C = 0 \quad (8\text{b})$$

$$M_{x,xx} + 2M_{sx,sx} + M_{s,ss} + 6K^2CN_s/\rho = -6K^4C^2 \quad (8\text{c})$$

The nondimensional effective shear stresses, which arise in the minimum principle and are to be used as natural boundary conditions, are

$$(Q_x)_{\text{eff}} = (M_{x,x} + 2M_{sx,s})/6K^2C \quad (9\text{a})$$

$$(N_{sx})_{\text{eff}} = N_{sx} - \delta M_{sx}/3K^2C\rho \quad (9\text{b})$$

where $(Q_x)_{\text{eff}}$ and $(N_{sx})_{\text{eff}}$ have been nondimensionalized by equations similar to Eq. (2b).

The system of equilibrium equations as well as the following compatibility conditions can also be obtained by specializing equations presented by Novozhilov[13] for arbitrary shells referred to curvature coordinates:

$$\varkappa_{s,x} - \varkappa_{sx,s} = 0 \quad (10\text{a})$$

$$\varkappa_{x,s} - \varkappa_{sx,x} + \delta(e_{sx,x} - e_{x,s})/\rho = 0 \quad (10\text{b})$$

$$e_{s,xx} - e_{sx,sx} + e_{x,ss} + \varkappa_x/\rho = 0 \quad (10\text{c})$$

The preceding system of differential equations can be written in a complex form by using the static-geometric analogy introduced into shell theory by Gol'denveizer.[12] Following Naghdi,[14] this is accomplished by introducing the following complex functions:

$$P_x = N_x + i\varkappa_s/2K^2,\ P_s = N_s + i\varkappa_x/2K^2 \quad (11\text{a,b})$$

$$P_{sx} = N_{sx} - i\kappa_{sx}/2K^2,\ R_x = M_x + 3iCe_s \quad (11c,d)$$

$$R_s = M_s + 3iCe_x,\ 2R_{sx} = 2M_{sx} - 3iCe_{sx} \quad (11e,f)$$

where $i = (-1)^{1/2}$. The equations of compatibility, Eqs. (10), are now multiplied by $i/2K^2$ and the results added to the equilibrium equations, Eqs. (8). This yields, upon use of Eqs. (11),

$$P_{x,x} + P_{sx,s} = 0 \quad (12a)$$

$$P_{s,s} + P_{sx,x} - \delta(2R_{sx,x} + R_{s,s})/6\rho K^2C = 0 \quad (12b)$$

$$R_{x,xx} + 2R_{sx,sx} + R_{s,ss} + 6K^2CP_s/\rho = -6K^4C^2 \quad (12c)$$

The real and imaginary parts of the preceding equations yield the equilibrium equations and compatibility equations, respectively.

The quantities P_x, P_s, P_{sx} and R_x, R_s, R_{sx} are not all independent of each other but are related as follows through Hooke's Laws, Eqs. (3):

$$R_x = 3iC(P_s - \nu\bar{P}_x),\ R_s = 3iC(P_x - \nu\bar{P}_s) \quad (13a,b)$$

$$R_{sx} = -3iC(P_{sx} + \nu\bar{P}_{sx}) \quad (13c)$$

where a bar over a quantity denotes its complex conjugate.

It is convenient in the present analysis to introduce complex functions Φ and Ψ defined such that

$$\Phi = w + i\varphi,\ \Psi = v + i\psi \quad (14a,b)$$

where φ and ψ satisfy

$$N_x = -(1/2K^2)(\varphi_{,s} + \delta\psi/\rho)_{,s}, \qquad N_s = -(1/2K^2)\varphi_{,xx} \quad (15a,b)$$

$$N_{sx} = (1/2K^2)(\varphi_{,s} + \delta\psi/\rho)_{,x} \quad (15c)$$

The use of Eqs. (6d–f, 11a–c, 14, and 15) shows that P_x, P_s, P_{sx} are related to Φ and Ψ by

$$P_x = (i/2K^2)(\Phi_{,s} + \delta\Psi/\rho)_{,s},\ P_s = (i/2K^2)\Phi_{,xx} \quad (16a,b)$$

$$P_{sx} = -(i/2K^2)(\Phi_{,s} + \delta\Psi/\rho)_{,x} \quad (16c)$$

Thus Eq. (12a) is satisfied identically. Equations (13) and (16) are used to reduce Eqs. (12b,c) to the following two equations for the two unknowns Φ and Ψ:

$$i(\Psi_{,s} - \Phi/\rho)_{,xx} + (1/2K^2)[\Phi_{,xx} + \nu(\bar{\Phi}_{,s} + \delta\bar{\Psi}/\rho)_{,s}]_{,xx} = 2K^4C \quad (17a)$$

$$\nabla^4\Phi + \delta\nabla^2(\Psi/\rho)_{,s} + \delta(\Psi/\rho - \nu\Psi/\rho)_{,xxs} - 2iK^2(\Phi/\rho)_{,xx} = 4K^6C \quad (17b)$$

where ∇^2 is the Laplacian operator and ∇^4 is the Biharmonic operator, i.e.,

$$\nabla^2(\ ) = (\ )_{,xx} + (\ )_{,ss} \quad (18a)$$

$$\nabla^4(\ ) = (\ )_{,xxxx} + 2(\ )_{,xxss} + (\ )_{,ssss} \quad (18b)$$

For a typical bay of unsupported length L the appropriate boundary and connecting conditions between the ring and shell are

$$u_{,s}(\pm x_R,s) = 0,\ w_{,x}(\pm x_R,s) = 0 \quad (19a,b)$$

$$v(\pm x_R,s) = V(s),\ w(\pm x_R,s) = W(s) \quad (19c,d)$$

$$\frac{1}{\pi}\int_0^\pi N_x ds = -\frac{K^2C}{2}\frac{A^*}{\pi r_0^2} \quad (19e)$$

where V and W are the circumferential and radially inward displacement components of the ring at the line of contact between ring and shell, A^* is the area enclosed by the oval, and x_R is the value of x at the ring, i.e.,

$$x_R = L/2r_0 \quad (20)$$

Equations (19a,b) permit no out-of-plane warping or twisting of the ring, Eqs. (19c,d) assure identical deformations of the ring and the shell, and for singly symmetric shells Eq. (19e) equates the total end load due to the hydrostatic pressure q_0 to the resultant axial force in the shell at the ends $x = \pm x_R$.

Equations (19c,d) imply that the circumferential strain of both ring and shell are identical along the line of contact, i.e.,

$$e_s(x_R,s) = \epsilon(s) \quad (21)$$

where

$$\epsilon(s) = dV/ds - W/\rho \quad (22)$$

Thus, Eq. (21) can be used to replace either of Eqs. (19c,d).

The circumferential and radially inward components of the interaction load on the ring, S and Z, respectively, are equal to twice the effective in-plane and transverse shears in the shell, because both the section of shell to the left and to the right of the ring interact with the ring. Therefore, in nondimensional form

$$S = -2T_R/K^2C,\ Z = -2Q_R/K^2C \quad (23a,b)$$

where

$$T_R(s) = (N_{sx})_{\text{eff}}(x_R,s),\ Q_R(s) = (Q_x)_{\text{eff}}(x_R,s) \quad (24a,b)$$

An analysis of nonuniform oval rings appropriate for the present application is presented in Ref. 8.

Asymptotic Solution for Shell

In accordance with Refs. 9–11, it is now assumed that the solution has two distinct characteristic lengths. The interior solution has a nondimensional characteristic length of order K and the edge solution has a nondimensional characteristic length of order $1/K$. The total solution is the sum of interior solution, the edge solution, and a particular integral.

In the asymptotic solution of Ref. 4 only the leading term of a given quantity was retained in the final solution. This caused results for the radial displacement to be inaccurate for shells having circular or nearly circular cross sections. In the present analysis certain higher order terms will be retained in order to insure that the solution is exact for circular cylindrical shells.

In both the interior and the edge solutions the governing partial differential equations are the homogeneous parts of Eqs. (17). The homogeneous part of Eq. (17a) is integrated twice with respect to x, and the functions of integration are set to zero since they do not have the characteristic exponential behavior of either the interior or edge solution.

A. Interior Solution

The interior solution, identified by a superscript i, varies slowly in the axial direction. A new independent variable given by

$$\beta = x/K \quad (25)$$

is substituted into the homogeneous parts of Eqs. (17), yielding

$$(\Phi^i_{,s} + \delta\Psi^i/\rho)_{,sss} - 2i(\Phi^i/\rho)_{,\beta\beta} + (1/K^2)[2(\Phi^i_{,s} + \delta\Psi^i/\rho) - \delta\nu\bar{\Psi}^i/\rho]_{,\beta\beta s} + (1/K^4)\Phi^i_{,\beta\beta\beta\beta} = 0 \quad (26a)$$

$$i(\Psi^i_{,s} - \Phi^i/\rho) + (\nu/2K^2)(\bar{\Phi}^i_{,s} + \delta\bar{\Psi}^i/\rho)_{,s} + (1/2K^4)\Phi^i_{,\beta\beta} = 0 \quad (26b)$$

It is assumed that differentiation with respect to either β or s does not affect the order of magnitude of any of the terms. The functions Φ^i and Ψ^i are assumed as

$$\Phi^i = \Phi^i_0 + \Phi^i_2/K^2 + \ldots,\ \Psi^i = \Psi^i_0 + \Psi^i_2/K^2 + \ldots \quad (27a,b)$$

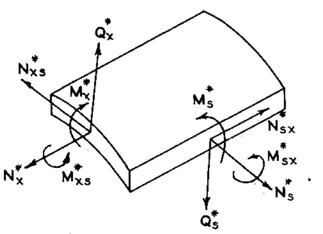

Fig. 2 Stress convention for shell.

where

$$\Psi_0{}^i = \sum_{j=0,2}^{\infty}\left(\frac{\beta^j}{j!}\right)A_j(s),\ \Psi_2{}^i = \sum_{j=0,2}^{\infty}\left(\frac{\beta^j}{j!}\right)B_j(s) \quad (28a,b)$$

$$\Psi_4{}^i = \sum_{j=0,2}^{\infty}\left(\frac{\beta^j}{j!}\right)C_j(s) \quad (28c)$$

and A_j, B_j, C_j are complex functions of s. Note that Ψ^i has been expanded as a power series in both β and $1/K^2$. The leading terms in the above series can be expected to yield accurate numerical results for short ($\beta \ll 1$), thin ($1/K^2 \ll 1$) shells. If Eqs. (27) and (28) are substituted into Eqs. (26) and the coefficients of like powers of β and $1/K^2$ are equated to zero, the following recursive relations are obtained for A_j and B_j:

$$A_j = -i[(\rho A_{j-2}')' + \delta A_{j-2}/\rho]''/2; j = 2,4, \ldots \quad (29a)$$

$$B_j = -(i/2)[(\rho B_{j-2}')' + \delta B_{j-2}/\rho]'' - (\nu/4)\{\rho[(\rho \bar{A}_{j-2}') + \delta \bar{A}_{j-2}/\rho]'\}''' - (i/2)\{2[(\rho A_j')' + \delta A_j/\rho] - \nu[(\rho \bar{A}_j')' + 2\delta \bar{A}_j/\rho]\} \quad j = 2,4, \ldots \quad (29b)$$

where a prime indicates ordinary differentiation with respect to s. Thus, if A_0, B_0, C_0 are arbitrary complex constants, the functions Φ^i and Ψ^i become

$$\Psi^i = A_0 + (x^2/2K^2)A_2 + (x^4/24K^4)A_4 + (1/K^2)B_0 + (x^2/2K^4)B_2 + (1/K^4)C_0 + \ldots \quad (30a)$$

$$\Phi^i = \rho A_0' + (x^2/2K^2)\rho A_2' + (x^4/24K^4)\rho A_4' + (1/K^2)\rho B_0' - (i\nu\rho/2K^2)[(\rho \bar{A}_0')' + \delta \bar{A}_0/\rho]' + (x^2/2K^4)\{\rho B_2' - (i\nu\rho/2)[(\rho \bar{A}_2')' + \delta \bar{A}_2/\rho]'\} + (1/K^4)\rho C_0' - (i\nu\rho/2K^4)[(\rho \bar{B}_0')' + \delta \bar{B}_0/\rho]' + (\nu^2\rho/4K^4)\{\rho[(\rho A_0')' + \delta A_0/\rho]'\}'' - (i\rho^2/2K^4)A_2' + \ldots \quad (30b)$$

B. Complete Interior Solution

The sum of the interior solution and an appropriate particular integral which satisfies the nonhomogeneous governing partial differential equations is termed the complete interior solution. Suitable particular integrals, identified by the superscript P, can be obtained by assuming that

$$u^P(x,s) = c_1 x,\ v^P(x,s) = v^P(s),\ w^P(x,s) = w^P(s) \quad (31a,b,c)$$

where c_1 is a constant. In Ref. 15 it is shown that under the preceding assumptions, the particular integral takes the form

$$\Psi^P = K^6\tilde{v}^P + K^2C(1-\delta)\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right)\int_0^s \frac{(1-\rho)ds}{\rho} - \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\tilde{Q} + \cos\theta \int_0^s \tilde{e}^P \cos\theta ds - \sin\theta \times \int_s^{\pi/2} \tilde{e}^P \sin\theta ds\right\} + iK^4C\rho\left[2\nu\left(c_2\int_0^s \cos ds\theta - \int_0^s \tilde{N}ds + \frac{s}{\pi}\int_0^{\pi}\tilde{N}ds\right) + x^2\left(\frac{c_2}{\rho}\sin\theta + \tilde{N}'\right)\right] \quad (32a)$$

$$\Phi^P = K^6 \rho \tilde{v}'^P + K^2C(1-\delta)\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right) + \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\tilde{N} + \sin\theta\int_0^s \tilde{e}^P\cos\theta ds + \cos\theta\int_s^{\pi/2}\tilde{e}^P \sin\theta ds\right\} + iK^4C\left[x^2 + \frac{1}{2}\frac{A^* s^2}{\pi r_0{}^2} + \delta x^2(-1 + c_2\cos\theta - \tilde{N})\right] \quad (32b)$$

where

$$\tilde{N} = -\sin\theta\int_0^s \cos\theta ds - \cos\theta\int_s^{\pi/2}\sin\theta ds \quad (33a)$$

$$\tilde{Q} = -\cos\theta\int_0^s \cos\theta ds + \sin\theta\int_s^{\pi/2}\sin\theta ds \quad (33b)$$

$$\tilde{e}^P = (1-\nu^2)c_2\cos\theta - (1-\nu^2)\tilde{N} - (\nu^2/\pi)\int_0^{\pi}\tilde{N}ds \quad (33c)$$

and θ is the angle between the axis of symmetry and an outer normal to the oval line of contact between ring and shell so that

$$1/\rho = d\theta/ds \quad (34)$$

Also, $\tilde{v}^P$ is a real function of s and c_2 is a real constant. Exact formulas for $\tilde{v}^P$ and c_2 appear in Ref. 15. Their formulas have been omitted here because they are lengthy and do not contribute to the final results. Herein it is only necessary to state that both quantities are of order of magnitude unity and that c_2 vanishes when the shell and ring are doubly symmetric.

It is known[1,4] that the total solutions for both Φ and Ψ are of order K^4 and have a slow variation in the axial direction. The particular solution contains terms of orders K^6, K^4, and K^2. To insure a slowly varying result of the proper magnitude the term of order K^6 must be cancelled by the interior solution. This is accomplished by setting A_0 equal to the negative of the K^6th part of v^P. The resulting combination of interior and particular solutions yields the complete interior solution.

The preceding considerations together with Eq. (32a) imply that

$$A_0 = -K^6\tilde{v}^P \quad (35)$$

Furthermore, by examining Eq. (32b) it can be concluded that the K^6th part of w^P is equal to the negative of $\rho A_0'$. These statements upon consideration of Eqs. (3, 6, 8b, 8c, 31, 32) imply that to leading terms

$$\varkappa_s{}^P = -[(\rho A_0')' + \delta A_0/\rho]' \quad (36a)$$

$$(\rho\varkappa_s{}^{P''})' + \delta\varkappa_s{}^{P'}/\rho = 4K^6C\rho' \quad (36b)$$

Eqs. (36) are useful in simplifying the recursive relations, Eqs. (29).

The interior solution, Eqs. (30), is added to the particular integral Eqs. (32) to obtain the following complete interior solution:

$$\Psi^i + \Psi^P = \frac{B_0}{K^2} + iK^4C\rho\left[2\nu\left(c_2\int_0^s \cos\theta ds - \int_0^s \tilde{N}ds + \frac{s}{\pi}\int_0^{\pi}\tilde{N}ds\right) + x^2\left(\frac{c_2}{\rho}\sin\theta + \tilde{N}'\right)\right] + K^2C(1-\delta) X\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right)\int_0^s \frac{(1-\rho)ds}{\rho} - \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\tilde{Q} + \cos\theta\int_0^s \tilde{e}^P\cos\theta ds - \sin\theta\int_s^{\pi/2}\tilde{e}^P\sin\theta ds\right\} + \frac{C_0}{K^4} + \frac{ix^2}{4K^2}\varkappa_s{}'^P + \frac{ix^2}{4K^4}b_0' + \frac{x^2}{8K^4}\left[4K^6C\rho'(2+\nu) + \nu\delta\frac{\varkappa_s{}'^P}{\rho}\right] + \frac{1}{24}K^2Cx^4\rho''' \quad (37a)$$

$$\Phi^i + \Phi^P = \frac{\rho B_0'}{K^2} + \frac{i\nu}{2K^2}\rho\kappa_s{}^P + iK^4C\left[x^2 + \frac{1}{2}\frac{A^*s^2}{\pi r_0{}^2} + \delta x^2(-1 + c_2\cos\theta - \tilde{N})\right] + K^2C(1-\delta)\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right) + \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\tilde{N} + \sin\theta\int_0^s \bar{e}^P\cos\theta ds + \cos\theta\int_s^{\pi/2}\bar{e}^P\sin\theta ds\right\} + \frac{\rho C_0'}{K^4} + \frac{\rho^2\kappa_s{}''^P}{4K^4} + \frac{i\nu\rho}{2K^4}\bar{b}_0 + \frac{ix^2}{4K^2}\rho\kappa_s{}''^P + \frac{ix^2\rho}{4K^4}b_0'' + \frac{x^2\rho}{8K^4}\left[8K^6C\rho'' + \delta\nu\left(\frac{\kappa_s{}'^P}{\rho}\right)'\right] + \frac{1}{24}K^2Cx^4\rho\rho^{IV} \quad (37b)$$

where

$$b_0 = -[(\rho B_0')' + \delta B_0/\rho + (i\nu/2)(\rho\kappa_s{}^P)']' \quad (38)$$

Equations (29, 35, and 36) were used to simplify Eqs. (37). Also, in writing Eqs. (37), sufficient terms have been retained to insure that the leading term of any quantity will not be lost by differentiation.

Equations (37) can now be used in conjunction with Eqs. (3, 6, 9, 14, and 15) to obtain the stresses, strains, and displacements for the complete interior solution. Expressions for some important quantities are listed below:

$$u^i + u^P = \frac{x}{2K^4}Im(b_0) + K^2Cx\left[\nu\rho - \frac{1}{2}\frac{A^*}{\pi r_0{}^2} - \nu\delta\left(c_2\cos\theta - \tilde{N} + \frac{1}{\pi}\int_0^\pi \tilde{N}ds\right)\right] \quad (39a)$$

$$v^i + v^P = Re\left(\frac{B_0}{K^2}\right) + K^2C(1-\delta)\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right) \times \int_0^s (1-\rho)ds/\rho - \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\tilde{Q} + \cos\theta\int_0^s \bar{e}^P\cos\theta ds - \sin\theta\int_s^{\pi/2}\bar{e}^P\sin\theta ds\right\} \quad (39b)$$

$$w^i + w^P = Re\left(\frac{\rho B_0'}{K^2}\right) + K^2C\left[1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2} + \rho(\rho - 1)\right] + \delta K^2C\left\{-\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2} - \rho\right) + \left(\rho + \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2}\right)\tilde{N} - c_2\rho\cos\theta + \sin\theta\int_0^s \bar{e}^P\cos\theta ds + \cos\theta\int_s^{\pi/2}\bar{e}^P\sin\theta ds\right\} + \frac{\nu\rho}{2K^4}Im(b_0) \quad (39c)$$

$$e_s{}^i + e_s{}^P = -K^2C\left[\rho - (\nu/2)(A^*/\pi r_0{}^2) - \nu^2\delta\left(c_2\cos\theta - \tilde{N} + \frac{1}{\pi}\int_0^\pi \tilde{N}ds\right)\right] - (\nu/2K^4)Im(b_0) \quad (39d)$$

$$N_x{}^i + N_x{}^P = Im(b_0)/2K^4 - K^2C\left[(1/2)(A^*/\pi r_0{}^2) + \nu\delta\left(c_2\cos\theta - \tilde{N} + \frac{1}{\pi}\int_0^\pi \tilde{N}ds\right)\right] \quad (39e)$$

$$N_s{}^i + N_s{}^P = -K^2C\rho \quad (39f)$$

$$(N_{sx}{}^i)_{\text{eff}} + (N_{sx}{}^P)_{\text{eff}} = N_{sx}{}^i + N_{sx}{}^P = K^2Cx\rho' \quad (39g)$$

$$M_x{}^i + M_x{}^P = 3\nu CRe(b_0)/2K^4 \quad (39h)$$

$$M_s{}^i + M_s{}^P = 3CRe(b_0)/2K^4 \quad (39i)$$

Equations (33, 36, and 38) were used to simplify the preceding results.

C. Edge Solution

The edge solution identified by a superscript e has a short characteristic length. A new independent variable α given by

$$\alpha = Kx \quad (40)$$

is substituted into the homogeneous parts of Eqs. (17) yielding

$$\Phi^e{}_{,\alpha\alpha\alpha\alpha} - 2i(\Phi^e/\rho)_{,\alpha\alpha} + (1/K^2)[2(\Phi_{,s}{}^e + \delta\Psi^e/\rho) - \nu\delta\bar{\Psi}^e/\rho]_{,\alpha\alpha s} + (1/K^4)(\Phi_{,s}{}^e + \delta\Psi^e/\rho)_{,sss} = 0 \quad (41a)$$

$$\Phi^e{}_{,\alpha\alpha} + 2i(\Psi_{,s}{}^e - \Phi^e/\rho) + (\nu/K^2)(\bar{\Phi}_{,s}{}^e + \delta\Psi^e/\rho)_{,s} = 0 \quad (41b)$$

It is assumed that differentiation with respect to either α or s does not affect the order of magnitude of any of the terms.

The functions Φ^e and Ψ^e are expanded in the following power series in $1/K^2$

$$\Phi^e = \Phi_0{}^e + \Phi_2{}^e/K^2 + \ldots \quad (42a)$$

$$\Psi^e = (1/K^2)(\Psi_0{}^e + \Psi_2{}^e/K^2 + \ldots) \quad (42b)$$

where all the $\Phi_j{}^e$ and $\Psi_j{}^e$ are of the same order of magnitude, hence, Ψ^e is of smaller magnitude than Φ^e. Substituting Eqs. (42) into Eqs. (41) and equating the coefficients of like powers of $1/K^2$ to zero yields, for $j = 0,2$

$$\Phi^e{}_{0,\alpha\alpha} - 2i\Phi_0{}^e/\rho = 0 \quad (43a)$$

$$(\Phi^e{}_{2,\alpha\alpha} - 2i\Phi_2{}^e/\rho)_{,\alpha\alpha} + 2\Phi^e{}_{0,\alpha\alpha ss} = 0 \quad (43b)$$

$$\Phi^e{}_{2,\alpha\alpha} - 2i\Phi_2{}^e/\rho + (2i\Psi_0{}^e + \nu\bar{\Phi}_{0,s}{}^e)_{,s} = 0 \quad (43c)$$

The solution of Eq. (43a) that is symmetric in x is

$$\Phi_0{}^e = A(s)\cosh\lambda(s)x \quad (44)$$

where

$$\lambda(s) = (1 + i)K/(\rho)^{1/2} \quad (45)$$

and $A(s)$ is a complex function to be determined by the boundary conditions.

The function $\Psi_0{}^e$ is determined by substituting Eq. (43c) into Eq. (43b) and integrating once with respect to α and twice with respect to s, yielding

$$\Psi_0{}^e = -(i/2)(2\Phi_0{}^e - \nu\bar{\Phi}_0{}^e)_{,s} \quad (46)$$

The functions of integration have been set equal to zero since they do not have the proper characteristic behavior of an edge zone solution.

Equations (44–46) can be used in conjunction with Eqs. (3, 6, 9, 14, and 15) to obtain the stresses, strains, and displacements for the edge solution. The leading terms of some important quantities are listed below:

$$u^e = \nu\varphi_{0,x}{}^e/2K^2,\ v^e = v_0{}^e = (2+\nu)\varphi_{0,s}{}^e/2K^2 \quad (47a,b)$$

$$w^e = w_0{}^e,\ e_s{}^e = -\varphi^e{}_{0,xx}/2K^2,\ N_x{}^e = -\varphi^e{}_{0,ss}/2K^2 \quad (47c,d,e)$$

$$N_s{}^e = -\varphi^e{}_{0,xx}/2K^2,\ (N_{sx}{}^e)_{\text{eff}} = N_{sx}{}^e = \varphi^e{}_{0,sx}/2K^2 \quad (47f,g)$$

$$M_x{}^e = -3Cw^e{}_{0,xx}/2K^2,\ M_s{}^e = -3\nu Cw^e{}_{0,xx}/2K^2 \quad (47h,i)$$

$$M_{sx}{}^e = -3C(1-\nu)w^e{}_{0,sx}/2K^2,\ (Q_x{}^e)_{\text{eff}} = -w^e{}_{0,xxx}/4K^4 \quad (47j,k)$$

The relationship

$$-\rho w^e{}_{0,xxx} = 2K^2\varphi_{0,x}{}^e \quad (48)$$

can be readily obtained from the solution, Eq. (44). Examination of Eqs. (47g, 47k, and 48) shows the following relationship, which is valid to the leading terms retained:

$$(N_{sx}{}^e)_{\text{eff}} = [\rho(Q_x{}^e)_{\text{eff}}]_{,s} \quad (49)$$

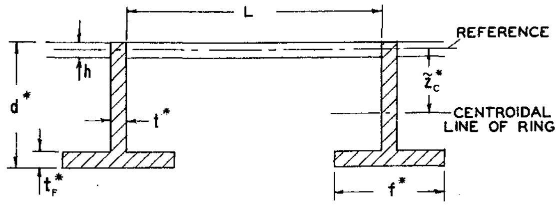

Fig. 3 Longitudinal cross section of ring reinforced cylinder.

Solution for Ring-Reinforced Cylinder

The complete solution for a ring-reinforced cylinder is obtained by combining the ring solution of Ref. 8 with the asymptotic shell solution. The asymptotic shell solution is the sum of the complete interior solution (Sec. B) and the edge solution (Sec. C). The ring solution of Ref. 8 is used in applying the boundary conditions, Eqs. (19, 21, and 23).

As previously noted, the total solutions for both Φ and Ψ are of order K^4 and have a slow variation in the axial direction. Therefore, the slowly varying complete interior solution must be of order K^4 and the rapidly varying edge solution must be of order K^2. These considerations together with Eqs. (37) and (44) imply that $B_0(s)$ is of order K^6 and $A(s)$ is of order K^2. In addition, it can readily be shown from Eqs. (3d, 3f, 9a, 37, and 47k) that

$$(Q_x{}^i)_{\text{eff}} + (Q_x{}^P)_{\text{eff}} \ll (Q_x{}^e)_{\text{eff}} \tag{50}$$

Equations (47k, 48, and 50) are used to write Eq. (24b) as

$$Q_R = \varphi_{,x}{}^e(x_R,s)/2K^2\rho = Im[\Phi_{,x}{}^e(x_R,s)]/2K^2\rho \tag{51}$$

Similarly, it can be shown that $w_{,x}$ also depends only on the edge solution. Therefore, for a leading term solution, the boundary condition (19b) becomes

$$w_{,x}{}^e(x_R,s) = Re[\Phi_{,x}{}^e(x_R,s)] = 0 \tag{52}$$

The combination of Eqs. (51) and (52) yields the following boundary condition for the edge solution:

$$\Phi_{,x}{}^e(x_R,s) = 2iK^2\rho Q_R \tag{53}$$

The arbitrary function $A(s)$ in the edge solution can be expressed in terms of Q_R by substituting Eq. (44) in Eq. (53). The edge solution then becomes

$$\Phi^e = \lambda\rho^2 Q_R \cosh\lambda x/\sinh\lambda x_R \tag{54}$$

Consideration of Eqs. (39e) and (47e) shows that $N_x{}^e$ is negligible compared to $N_x{}^i + N_x{}^P$; hence, the boundary condition (19e) upon use of Eq. (39e) becomes

$$\int_0^\pi Im(b_0)ds = 0 \tag{55}$$

where use was made of the following condition, which is due to the singly symmetric geometry:

$$\int_0^\pi \cos\theta ds = 0 \tag{56}$$

The axial displacement boundary condition, Eq. (19a) implies that at $x = x_R$ the axial displacement u is independent of s and thus is a constant, say c_3. Using Eqs. (39a, 47a, and 54) implies

$$c_3 = \nu\rho Q_R + \frac{x_R}{2K^4} Im(b_0) + K^2 C x_R \left\{ \nu\rho - \frac{1}{2}\frac{A^*}{\pi r_0{}^2} - \nu\delta\left[c_2\cos\theta - \tilde{N} + \frac{1}{\pi}\int_0^\pi \tilde{N}ds\right]\right\} \tag{57}$$

Equations (56) and (57) are solved simultaneously to obtain c_3 and $Im(b_0)$:

$$c_3 = -\frac{K^2C}{2}\left[\frac{\nu}{\pi}\int_0^\pi \rho P(s)ds + x_R\frac{A^*}{\pi r_0{}^2}\right] \tag{58a}$$

$$Im(b_0) = K_6 C\nu\left\{\frac{1}{x_R}\left[\rho P - \frac{1}{\pi}\int_0^\pi \rho P ds\right] + 2\delta\left[c_2\cos\theta - \tilde{N} + \frac{1}{\pi}\int_0^\pi \tilde{N}ds\right]\right\} \tag{58b}$$

where use was made of Eq. (56), and

$$P(s) = -2(Q_R/K^2C + x_R) \tag{59}$$

All elements of the shell solution are now either known, or known in terms of the undetermined functions $P(s)$ and $Re(B_0)$. These functions can be determined as follows by satisfying the interaction conditions between the shell and the ring.

The load on the ring acts at the ring-shell contact line, herein called the reference line, and is due to the interaction loads S and Z as well as to the direct action of the external pressure q_0. The loading may be written in terms of P as follows by utilizing Eqs. (23, 24, 39g, 47g, 47k, 49, 50, and 59):

$$S_T = S = (\rho P)', \quad Z_T = Z + t = P + 2x_R + t \tag{60a,b}$$

where S_T and Z_T, respectively, are the total unit nondimensional circumferential and radial loads on the ring, and t is the width of the ring at the reference line, nondimensionalized with respect to r_0.

The function P is determined by enforcing Eq. (21), i.e., by requiring the circumferential strain acting on fibers along the reference line to be the same in both the ring and the shell. The circumferential strain acting on the reference line of the ring can be obtained from Ref. 8. For the loading given by Eqs. (60), this strain assumes the following functional form:

$$\epsilon(s) = f_1(s) + f_2(s)\rho P + f_3(s)\int_0^\pi f_5(s)\rho P ds + f_4(s)\int_0^\pi f_6(s)\rho P ds \tag{61}$$

where $f_1(s), \ldots, f_6(s)$ are singly symmetric functions of s. Also, the last integral in Eq. (61) vanishes for doubly symmetric ring-shell structures.

The circumferential strain in the shell is determined from Eqs. (39d, 45, 47d, 54, 58b, and 59). The result at $x = x_R$ is

$$e_s(x_R,s) = -\frac{K^2C}{2}\rho P\left[\frac{\nu^2}{x_R} - Re(\lambda\coth\lambda x_R)\right] + \frac{K^2C}{2}\frac{\nu^2}{\pi x_R}\int_0^\pi \rho P ds + K^2C\left[\rho x_R Re(\lambda\coth\lambda x_R) + \frac{\nu}{2}\frac{A^*}{\pi r_0{}^2} - \rho\right] \tag{62}$$

The use of Eqs. (21, 61, and 62) yields the following integral equation for P:

$$\rho P = F_1(s) + F_2(s)J(\rho P) + F_3(s)J(f_5\rho P) + F_4(s)J(f_6\rho P) \tag{63}$$

where

$$F_0(s) = -f_2(s) - (K^2C/2)[\nu^2/x_R - Re(\lambda\coth\lambda x_R)] \tag{64a}$$

$$F_0(s)F_1(s) = f_1(s) - K^2C[\rho x_R Re(\lambda\coth\lambda x_R) + (\nu/2)(A^*/\pi r_0{}^2) - \rho] \tag{64b}$$

$$F_0(s)F_2(s) = -(K^2C/2)(\nu^2/x_R) \tag{64c}$$

$$F_0(s)F_3(s) = f_3(s)/\pi \qquad F_0(s)F_4(s) = f_4(s)/\pi \tag{64e}$$

and J is an integral operator defined as

$$J(\quad) = \frac{1}{\pi}\int_0^\pi (\quad)ds \tag{65}$$

The solution of Eq. (63) is obtained by first multiplying it by $1/\pi$, then by $f_5(s)/\pi$, and finally by $f_6(s)/\pi$. In each case the equations are integrated with respect to s from 0 to π. This yields three linear algebraic equations which can readily be solved for $J(\rho P)$, $J(f_5\rho P)$, and $J(f_6\rho P)$. The solution for P then follows from Eq. (63).

The only remaining undetermined function $Re(B_0)$, which was introduced in the interior solution, can be evaluated by using Eq. (19c). To this end, note that v^e is negligible compared to the sum $v^i + v^p$. Therefore, substituting Eq. (39b) into Eq. (19c) gives

$$Re\left(\frac{B_0}{K^2}\right) = V(s) - K^2C(1-\delta)\left(1 - \frac{\nu}{2}\frac{A^*}{\pi r_0^2}\right) \times \int_0^s (1-\rho)\frac{ds}{\rho} + \delta K^2C\left\{\frac{\nu}{2}\frac{A^*}{\pi r_0^2}\bar{Q} + \cos\theta\int_0^s \bar{e}^P\cos\theta ds - \sin\theta\int_s^{\pi/2}\bar{e}^P\sin\theta ds\right\} \tag{66}$$

where $V(s)$ is known from the ring solution of Ref. 8.

In principle all of the arbitrary functions have been determined; however, it is useful to develop the following special formula for $Re(b_0)$, which is valid to leading terms

$$Re(b_0) = -K^2[W' + \delta V/\rho]' \tag{67}$$

Again, $W(s)$ is known from the ring solution of Ref. 8.

The displacements and stress resultants for a thin, short, ring-reinforced oval cylindrical shell are presented below:

$$u(x,s) = \nu(\rho Q_R)Re\left[\frac{\sinh\lambda x}{\sinh\lambda x_R}\right] + K^2Cx\left[\nu\rho - \frac{1}{2}\frac{A^*}{\pi r_0^2} + \frac{\nu}{2x_R}\left(\rho P - \frac{1}{\pi}\int_0^\pi \rho P ds\right)\right] \tag{68a}$$

$$v(x,s) = V(s) \tag{68b}$$

$$w(x,s) = W(s) - \rho(\rho Q_R)Re[\lambda(\cosh\lambda x_R - \cosh\lambda x)/\sinh\lambda x_R] \tag{68c}$$

$$N_x(x,s) = -\frac{K^2C}{2}\left[\frac{A^*}{\pi r_0^2} - \frac{\nu}{x_R}\left(\rho P - \frac{1}{\pi}\int_0^\pi \rho P ds\right)\right] \tag{68d}$$

$$N_s(x,s) = -(\rho Q_R)Re[\lambda\cosh\lambda x/\sinh\lambda x_R] - K^2C\rho \tag{68e}$$

$$N_{sx}(x,s) = Re\{(\rho Q_R)'\sinh\lambda x/\sinh\lambda x_R + (\rho Q_R)\lambda\rho'(x_R\cosh\lambda x_R\sinh\lambda x - x\cosh\lambda x\sinh\lambda x_R)/(2\rho\sinh^2\lambda x_R)\} + K^2Cx\rho' \tag{68f}$$

$$M_x(x,s) = 3C(\rho Q_R)Im[\lambda\cosh\lambda x/\sinh\lambda x_R] + (3\nu C/2K^4)Re(b_0) \tag{68g}$$

Table 1 **Comparison of Donnell and Love Solutions for uniform doubly symmetric inside ring**[a]

Arc length, s	Midbay ($x/x_R = 0$) M_x D	M_x L	M_s D	M_s L	Ring ($x/x_R = 1$) M_x D	M_x L	M_s D	M_s L
0	127	135	−39	−12	−434	−425	−208	−180
$\pi/8$	95	98	−17	−7	−297	−294	−135	−124
$\pi/4$	21	19	17	7	−40	−43	−2	−12
$3\pi/8$	−23	−27	39	28	99	95	75	65
$\pi/2$	−27	−29	49	41	115	113	91	84

[a] D: Donnell accuracy ($\delta = 0$). L: Love accuracy ($\delta = 1$).

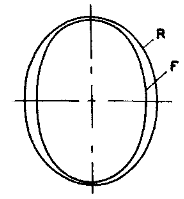

Fig. 4 Ring for $\zeta_2 > 0$. R—ring shell contact line (toe of ring), F—flange of ring.

$$M_s(x,s) = 3\nu C(\rho Q_R)Im[\lambda\cosh\lambda x/\sinh\lambda x_R] + (3C/2K^4)Re(b_0) \tag{68h}$$

$$M_{sx}(x,s) = 3C(1-\nu)Im\{(\rho Q_R)'\sinh\lambda x/\sinh\lambda x_R + (\rho Q_R)\lambda\rho'(x_R\cosh\lambda x_R\sinh\lambda x - x\cosh\lambda x\sinh\lambda x_R)/(2\rho\sinh^2\lambda x_R)\} \tag{68i}$$

where the quantities P, Q_R, and $Re(b_0)$ are given by Eqs. (63, 59, and 67), respectively. Formulas for the ring displacements V and W may be obtained from Ref. 8.

Application

The present asymptotic solution has been applied to a shell reinforced by tee-section rings (see Fig. 3). It is assumed that the depth d^* and flange width f^* are much larger than the flange thickness t_f^* and web thickness t^* (corresponding unstarred dimensions are nondimensionalized with respect to r_0). For simplicity the ring depth is assumed to vary circumferentially such that the nondimensional distance $\bar{z}_c$ between the reference line and the centroidal line is given by

$$\bar{z}_c = \bar{z}_0(1 + \zeta_j\cos js) \tag{69}$$

where for a singly symmetric circumferential variation $j = 1$, whereas, for a doubly symmetric variation $j = 2$. It is assumed that $-1 < \zeta_j < 1$ so that the depth of the tee-section is never zero. The condition $\zeta_j = 0$ corresponds to a uniform reinforcing ring for which $\bar{z}_c$ has the constant value $\bar{z}_0$. Inside or outside rings are distinguished by setting $\bar{z}_0 > 0$ or $\bar{z}_0 < 0$, respectively.

The reference line for the numerical examples considered is assumed to be a doubly symmetric oval for which the local curvature is given by

$$1/\rho = 1 + \xi\cos 2s \tag{70}$$

In order to avoid negative curvature, ξ must be restricted to the range $-1 \leq \xi \leq 1$. A detailed discussion of the geometry of such ovals appears in Ref. 7, where it is shown that

$$A^*/\pi r_0^2 = 1 - \xi^2/6 + \xi^4/240 + \ldots \tag{71}$$

The examples considered have doubly symmetric ($j = 2$), inside ($\bar{z}_0 > 0$) tee-sections with $\xi = 0.3891$, $x_R = 0.1309$, $f = 0.08071$, $h/r_0 = 0.01091$, $t = t_f = 0.004818$. With $\bar{z}_0 =$

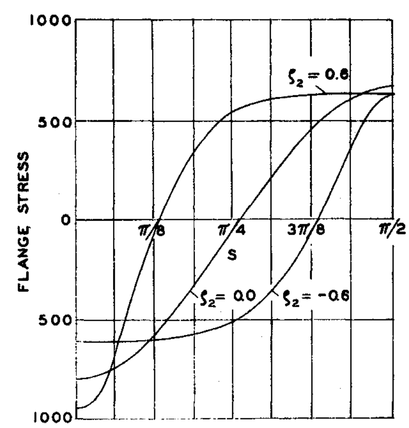

Fig. 5 Stress in flange of ring.

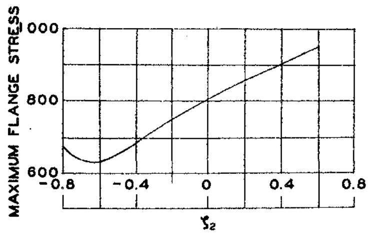

Fig. 6 Maximum stress in flange of ring vs ζ_2.

0.06545 and $\zeta_2 = 0$ this data corresponds to case 3 of Ref. 4. In the present work, rings having values of $\zeta_2 = 0, \pm 0.6$ are considered. The volume (hence the weight) of the rings is kept constant by adjusting $\bar{z}_0$ when changing ζ_2. A typical ring for $\zeta_2 > 0$ is depicted in Fig. 4. Negative values of ζ_2 correspond to a ring which is thicker at the minor axis than the major axis.

Discussion

The tracer constant δ was introduced into the equations in order to keep account of terms commonly omitted in Donnell type equations. However, after checking the solution, Eqs. (68), it can be seen that the only place where δ appears is through the function $Re(b_0)$ [see Eq. (67)]. Therefore, it may be concluded that all quantities in the asymptotic solution may be adequately described by the simple Donnell equations, with the possible exception of the axial bending moment M_x and circumferential bending moment M_s [since both Eq. (68g) for M_x and Eq. (68h) for M_s contain $Re(b_0)$]. Examining the results of Table 1 shows that Donnell theory can be used to predict the axial bending moment to a good degree of accuracy while giving only a fair indication of the circumferential bending moment. However, note that the circumferential bending moment is small compared to the axial bending moment.

It is shown in Ref. 16 that the maximum stress in the ring-shell structure is the stress in the flange of the ring. Figure 5 shows that for doubly symmetric inside rings the maximum flange stress can be reduced by adjusting ζ_2, i.e., by appropriately varying the depth while maintaining the volume constant. In Ref. 16 it is shown that a similar reduction also occurs for doubly symmetric outside rings.

Examination of Fig. 6 shows that for the cases considered $\zeta_2 = -0.6$ is an optimal value which minimizes the magnitude of the maximum flange stress. This value yields a structure having a maximum flange stress 21% lower than a structure having a uniform ring.

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